Exam Mathematical Models for the spread of infectious diseases (WMMA061-05)
Thursday February 1 2024, 11.45-13.45

This exam consists of 3 exercises all with subquestions. You can get 90 points in total and the grade for this written exam is obtained through

$$1 + \frac{\text{obtained points}}{10}.$$

The grade for this written exam contributes 60% to the final grade of the course.

Write your name and student number on every page you hand in and number the pages.

Justify your answers, in some cases partial answers may be worth points.

1. Consider a supercritical standard SIR epidemic in a homogeneously mixing population. Assume that initially there are n susceptible people, $m = \lfloor \log n \rfloor$ infectious people and no removed individuals. The per pair infectious contact rate is λ/n and the infectious periods are distributed as L, and assume $Var(L) < \infty$. Let $Z^{(n)}(\infty)$ be the number of ultimately removed people and $z^{(n)} = Z^{(n)}(\infty)/n$. Below you may use without proof that $z^{(n)}$ converges in probability to some real number $z > 0$ as $n \to \infty$.

a) Provide an identity determining z, and justify your answer (a correct heuristic justification is enough). (15pt)

b)Let $W : [-e^{-1}, \infty) \mapsto [-1, \infty)$ be the principal branch of the Lambert W function. That is,

$$W(y) := \{w \in [-1, \infty); we^w = y\},$$

where you may use without proof that for all $y \in [-e^{-1}, \infty)$, $W(y)$ indeed has exactly one element in $[-1, \infty)$.

Show that $z = 1 + \frac{W(-Re^{-R})}{R}$, where $R = \lambda\mathbb{E}[L]$. (5pt)

2. Consider a supercritical Markov SIR epidemic on a Configuration Model network of N vertices with expection of the degrees $\mathbb{E}[D] = \mu$ and finite second moment of the degrees $\mathbb{E}[D^2] = \mu_2$. Assume that the infectious contact rate per pair of neighbors is λ and the infectious periods are distributed as L, which is exponentially distributed with expectation $1/\gamma$.

a) Compute the basic reproduction number R_0^{CM} for this model. (10pt)

b) Compute the Malthusian parameter α^{CM} and express the basic reproduction number R_0^{CM} in terms of μ, μ_2, γ and α^{CM}. (10pt)

c) Consider a homogeneously mixing population of size N in which the contact rate per pair of individuals is λ/N and the infectious periods are distributed as L as above. Compute also for this model the Malthusian parameter α^{HM} and the basic reproduction number R_0^{HM}. Then write also for this model the basic reproduction number R_0^{HM} as a function of the Malthusian parameter α^{HM} and of γ.

Finally show that for given $\alpha = \alpha^{CM} = \alpha^{HM} > 0$ and γ, $R_0^{HM} \geq R_0^{CM}$. (15pt)

3. Consider an SIR epidemic Markov model for a large homogeneous closed population in which people form couples, are serial monogamous (i.e. a person is in at most one couple at a time) and can only transmit within a couple.

More specific: Consider a population of size N. Initially there are $N_1(0)$ single people and $N_2(0)$ couples of people in a relationship (So, $N_1(0)+2N_2(0) = N$). Existing couples break up at rate ν and two single people form a couple at per pair of singles rate κ/N. Initially there is 1 couple consisting of a susceptible and an infectious person, while all other people (single or part of a couple) are susceptible. Within couples contacts occur according to independent Poisson processes with intensity β. If the contact is between a susceptible and infectious person, the susceptible becomes infectious immediately. Infectious people recover (and become immune forever) independently at per person rate γ. The process is a Markov process.

a) What is the probability that the susceptible person in an SI couple becomes infected before the pair breaks up? (10pt)

b) Use a deterministic approximation to find $\hat{x}$, which is the asymptotic (as $N \to \infty$) fraction of people that is single. (10pt)

Hint: Let $N_1 = N_1(t)$ be the number of single people and $x = x(t) = N_1/N$. Find for which x the rate of pair formation is equal to the rate of breaking of pairs.

c) Consider a person that is just infected. Approximate the process (no justification needed) in such a way that this person when single, becomes part of a couple at rate $\kappa\hat{x}$. Provide the distribution of the number of couples this person is part of during their infectious period. (10pt)

d) Using the same approximation as for part c, provide an expression for the basic reproduction number R_0 if $N_1(0)/N = \hat{x}$, and justify your answer. (5pt)